The Peculiarites of Black Hole Entropy

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March 14, 2024

A sweet disorder in the dress
Kindles in clothes a wantonness;
A lawn about the shoulders thrown
Into a fine distraction;
An erring lace, which here and there
Enthrals the crimson stomacher;
A cuff neglectful, and thereby
Ribands to flow confusedly;
A winning wave, deserving note,
In the tempestuous petticoat;
A careless shoe-string, in whose tie
I see a wild civility:
Do more bewitch me, than when art
Is too precise in every part.

– Robert Herrick
"Delight in Disorder"

what is black hole entropy?

$$\frac{A}{4}$$

possibly plus corrections of order $\hbar$

all possibly true. . .

but not illuminating

Outline

entropy is weird

- almost every physical theory defines (or at least admits) an entropy
- in every one, with a single intriguing exception, entropy is fundamentally modal
- thus, several traditional problems become more poignant and difficult, *e.g.*
    - relation to arrow of time
    - interpretation of the Second Law
- recognition points to new avenues of attack on old problems? or further reason for despair?

what I mean when I say "entropy is modal":

1. the definition derives from a principle that itself is modal in character: phenomenological entropy from Clausius and Kelvin Postulates
2. OR the quantity is not an intrinsic property of single state, but counterfactual measure of how state would change were it transformed into standard reference state: Clausius entropy
3. OR the quantity is not intrinsic property of single state, but property of modally characterized class of states: Boltzmann entropy
4. OR the quantity is not intrinsic property of single system, but is property of modally characterized class of systems: Gibbs entropy
5. OR a combination of these: von Neumann entropy

(I do not understand the relations among these conditions, if any) intimately related:

1. entropy mediates no physical coupling between physical systems
2. $\Rightarrow$ *no such thing as an entropometer*
3. exemplification: von Neumann entropy in QM/QFT *is not an observable* (self-adjoint operator)

semi-classical black holes

$$S = \frac{A}{4}$$

- intrinsic property of single state of individual system
- no arguments needed for natural zero-point
- no counterfactuals, not probabilistic
- physical significance fixed by non-modal principles (First Law, Area Law, Generalized Second Law)
- $\Rightarrow$ *black hole entropy is not modal!*
- $\Rightarrow$ *there is an entropometer!*

why attribute entropy to black holes in the first place, besides formal analogies between laws of ordinary thermodynamics and of black hole thermodynamics?

most initially plausible way to explain what black hole entropy measures, and why it has such a property in the first place:

Hawking radiation and its well defined temperature
(Sorkin 1998; Preskill 1994)

temperature and entropy go together like Wurst und Senf!

or Sturm und Drang?

BUT—no real connection with Hawking radiation/Hawking temperature:

1. some kinds of entropy (*e.g.*, Shannon/information, which many think important here) defined for systems without temperature
2. indeed, Hawking radiation is strictly kinematic: needs only Lorentz metric with appropriate affine structure, nothing to do with dynamics (Visser 1998, 2003; Barceló et al. 2011)
3. but entropy is fundamentally dynamic—that we identify it with *one quarter* the area depends on the form of the EFE (Wald 1993)

Bekenstein's (1972; 1973; 1974) original motivation:

TO SAVE THE SECOND LAW!

still seems to me the best argument
(when BHM First Law invoked)

(we'll talk about this next week)

1. thermodynamical (phenomenological/Clausius)?
2. Boltzmann?
3. Gibbs?
4. von Neumann (entanglement)?
5. Shannon (information)?
6. holographic?
7. something else entirely?

thermodynamical

I claim: there had better be, at least, a good thermodynamical conception

- without a justification and understanding as a truly thermodynamical entropy, no real evidence in first place that black holes have appropriate SM
- string theory, loop quantum gravity, . . . , can count all "microstates" they want, but we need independent reason they're counting *physically relevant* states

in favor of thermodynamical:

1. GSL (Bekenstein 1972, 1973, 1974)
2. semi-classical black holes support construction of Carnot-like cycles (Kaburaki and Okamoto 1991; Curiel 2014; Bravetti et al. 2016; Prunkl and Timpson 2019)
3. speculative (crazy?) arguments that classical black holes + strictly classical matter jointly have well defined thermodynamics (Curiel 2014 – a few folks like it, including, from time to time, Curiel)

plenty of grounds for questioning, criticizing the last two

SM

but we want SM!

classical GR, nor even SCG, alone cannot provide a statistical construction:

- SCG still treats the black hole as an entity defined entirely by classical spacetime geometry
- no way to describe a black hole by physical attributes arising as gross statistical measures over underlying, more finely grained quantities
- any statistical accounting, therefore, must come from a theory attributing to classical geometry itself a description based on appropriate micro-structure
- presumably quantum in nature, underlying classical spacetime description of black holes

Boltzmann

a surprisingly common argument for Boltzmann-type (Sorkin 1983; Preskill 1994):[1]

1. Planck length + distinguished geometry provide natural coarse-graining: cover event horizon with Planck-area tiles
2. the horizon then carries some kind of information with density approximately one bit (0/1) per unit area
3. total number of configurations of the order of $N \approx 2^A \Rightarrow S := \log N \approx A \log 2$
4. *voilà!*

1. accident that same folks make this argument as claim Hawking radiation justifies interpretation of black hole entropy?

virtues:

1. minimal physical assumptions
2. largely independent of details of any theory of QG

nonetheless, I think this is a crappy argument

- what is the yes-no question?
- either not counting micro-states relevant to *dynamics* in any straightforward way
- or else *strong* and *unwarranted* assumption that fundamental degrees of freedom are binary (or at least strictly and uniformly bounded by a very small number)
- and, if latter, then *strong* and *unwarranted* assumption that such degrees of freedom can couple in right way with "higher level" degrees of freedom of ordinary matter (related to Page-time problem)

I have similar problems with many Boltzmannian "state-counting" arguments in weak-regime quantum gravity calculations

- Strominger and Vafa (1996) in string theory ("self-intersections of D-branes")
- Rovelli (1996) in loop quantum gravity ("ensemble/superposition of event horizon states")
- Dou and Sorkin (2003) in causal set theory ("causal links crossing event horizon")
- ...

begs the question by *assuming*:

1. that they are counting the dynamically relevant states,
2. *and* that counting measure is the appropriate measure—but counting measure is almost never correct in SM (always need some weighting)

Gibbs

I know of no arguments for or against Gibbs in the literature

to move beyond thermodynamical entropy in current epistemic state, I think Gibbs is most appropriate:

1. almost all proofs of GSL use it (only one I know that doesn't is 2-d string theory – so I'm unimpressed)
2. Gibbsian statistical mechanics is what one wants exactly when no secure knowledge of micro-degrees of freedom and micro-dynamics, only that system couples thermally to external systems
3. avoids Boltzmann worries about latching onto "right" physical micro-degrees of freedom/dynamics

entanglement entropy

"black hole entropy proportional to accounting of cross-horizon quantum field entanglement correlations" (Sorkin 1983)

virtues:

- supports derivation of SCEFE (Jacobson 2016)

demerits:

- species problem
- how can it explain increase in entropy when a classical entropic object, like Wheeler's infamous cup of tea, falls into BH?
- how can entanglement correlations across the horizon be sensitive in the right way to the cup's mass and *only* its mass?
- $\Rightarrow$ absolutely no reason for it to show up at the classical level in the Area Theorem (pure differential geometry)
- it may be that as area increases then entanglement entropy increases, but there is no reason to suspect the converse, and *that* is the relevant issue
- entanglement itself has *deep*, unresolved conceptual and foundational problems (Earman 2015)

holographic

“area of privileged null or spacelike surface in bulk proportional to von Neumann entropy of CFT on boundary” (Ryu and Takayanagi 2006; Hubeny et al. 2007; Engelhardt and Wall 2015)

virtues:

- cool and exciting

demerits:

- compelling derivations only in non-physical spacetimes
- holography has even less epistemic warrant than most things in SCG

something else entirely?

exotica, championed by small but vocal minorities:

1. Barrow entropy
2. Kaniadakis entropy
3. Rényi entropy
4. Sharma-Mittal entropy
5. Tsallis entropy

beyond the scope of this talk

- where does black hole entropy "live"? on the horizon or inside the black hole?
- in particular, does black hole entropy have anything to do with the state or dynamics of anything in the interior?
- that is to say: why is it proportional to *area* and not, as for all other types of physical system, *volume*?
- how one answers this question has clear and sharp relevance for how one will formulate and try to prove the GSL

(*locus classicus*: Jacobson et al. 2005) general considerations in favor of outside:

1. most natural explanation of the area law is that S lives on the horizon
2. No-Hair theorems $\Rightarrow$ the horizon itself is dissipative, not the interior
3. the one-way character of horizons affords relatively direct proofs of entropy non-decrease for coarse-grained dynamics (next week!)

standard arguments for outside:

1. interior of black hole can be "arbitrarily large" (*e.g.*, glue an FLRW spacetime into the interior of Schwarzschild)—can have unbounded interior microstates
2. interior of black hole is wildly out of equilibrium, but black hole First Law is appropriate for system in equilibrium, and entropy counts microstates contributing to macroscopic equilibrium state—only event horizon is "in equilibrium"
3. related: dynamics of interior are essentially one-way, no room for anything like ergodicity
4. assume Hawking effect unitary (at least until close to evaporation):
    - 4.1 then Hawking radiation correlated with interior degrees of freedom
    - 4.2 so number of interior states must remain large enough to store correlations; but if (*ex hypothesi*) interior is "normal", those states decrease much faster than A
    - 4.3 so entropy better not count interior states (related to Page-Time Problem)

but this all raises puzzles, perhaps favoring inside:

1. Bekenstein (“thermodynamical”) entropy is then smaller than accounting of *all* microstates associated with black hole (A + interior)—how is this SM?
2. how does the event horizon “keep a record” of the entropy/mass-energy of all matter that passes through it (without assuming exotic physics in low-energy regimes)?
3. how can a derivation of the Second Law ignore all that?

questions

1. many different entropies: how do they relate to each other? must we pick one as The One and True Black Hole Entropy?
2. how to interpret the entropy: subjective, objective? identity claims?
3. disanalogies with ordinary entropy—virtue or demerit?
4. $A/4$ requires prescience?
5. the enormity and discontinuity of $A/4$: why? how?
6. why no significant results in AQFT-CST?
7. why demand SM calculations rely on quantum gravitational degrees of freedom?
8. all QG programs derive $A/4$: trivalizes the evidential power of derivations? (or: "The 'I'm Always Right' Problem")
9. what may this all say about the nature of spacetime and matter and their inter-relations?

disanalogies

1. not modal; relatedly: entropometer!; and has natural zero-point, unlike entropy for other classical systems
2. scales with square of mass, not linearly (even worse in dS/AdS)
3. isentropic does not imply reversibility
4. every other kind of physical system: possible to lower entropy by throwing in mass-energy; not black holes
5. seems no way to decrease entropy of one black hole while increasing that of another, when they interact as a joint system
6. no way to split and recombine a black hole isentropically ("composition of systems")

can these teach us something about the form of a possible SM to ground black hole entropy (virtue)? or do they militate against interpreting $A/4$ as physical entropy (demerit)?

- these differences strongly suggest that extension of entropy to black holes, if correct, should modify and enrich understanding of entropy as physical quantity
- ALSO for temperature and heat
- $\Rightarrow$ analogous to how extension of those classical quantities to Maxwell fields did at end of 19th century

necessity of QSM?

- SM entropy of ordinary systems calculated using "classical particles" or, at most, "non-relativistic qm particles" (*e.g.*, asphalt heating from Sun's blackbody radiation), never from QFT
- in any event, even if we tried to use an SM based on QFT to calculate entropy of air in room, we couldn't do it without making approximations and idealizations that would essentially make all degrees of freedom "localized into something like classical/quantum particles" anyway
- why do—ought—black holes differ in this regard?

triviality?

- essentially every QG program, no matter how different, claims to have derived Bekenstein entropy...
- does it then depend on something that is not peculiar to any particular quantum gravity program?
- if so, then such derivations cannot provide *any confirmation* to any QG program
- indeed, it seems to become even weaker than a minimal consistency requirement
- only way to avoid: all QG programs share relevant structures and principles; but no good evidence for this

many entropies

1. there are many entropy-like quantities one can associate with a black hole in many different contexts, under many different representations (Wilkins 1979; Wall 2009)
2. there is no *a priori* reason to think that exactly one of them is The One and True Black Hole Entropy (Aristotelian essentialism)
3. we are latching on to different, albeit similar things, things which may, in classical contexts, be identified (though perhaps not), but here should not be until further argument is given

what, if anything, can this tell us?

intepretation

- standard classification of interpretations of entropy, *viz.*, objective versus subjective, inadequate: entropy can be observer-dependent, yet still appropriately objective, and that in 2 ways (think of determinations of spatial length in special relativity):

  1. depending on kinds of interaction physically possible for observer to have with given system
  2. intimately related: how observer individuates and identifies system and relevant degrees of freedom, meaning both:
     - 2.1 way state-space representation of total "system + environment" is decomposed
     - 2.2 how micro-degrees of freedom coarse-grained

  these difficulties compounded in SCG

- how can physical quantity with properties as *prima facie* different from ordinary entropy as spatial area be *identical* with it? what is criterion for and meaning of the identity claim? depends on one's view of inter-theoretic relations (Curiel 2024)

disanalogies

1. not modal; relatedly: entropometer!; and has natural zero-point, unlike entropy for other classical systems
2. scales with square of mass, not linearly (even worse in AdS)
3. isentropic does not imply reversibility
4. every other kind of physical system: possible to lower entropy by throwing in mass-energy; not black holes
5. seems no way to decrease entropy of one black hole while increasing that of another, when they interact as a joint system
6. no way to split and recombine a black hole isentropically ("composition of systems") at macro level

can these teach us something about the form of a possible SM to ground black hole entropy (virtue)? or do they militate against interpreting $A/4$ as physical entropy (demerit)?

- these differences strongly suggest that extension of entropy to black holes, if correct, should modify and enrich understanding of entropy as physical quantity
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prescience?

1. in order to know current location, and so size, of standard event horizon, one needs to know its entire future
2. $\Rightarrow S_B$ cannot be a well defined state function on a phase space for most commonly used black hole models

the enormity and discontinuity of $A/4$

entropy of solar-mass black hole $\sim 10^{20}$ larger than Hydrogen gas cloud of same mass

- why—*how*—does the entropy in the relevant spacetime region jump discontinuously by such a Gargantuan amount?
- since standard event horizon is global (prescience), how does the world restricted to a spacelike hypersurface *know* that it has formed?
- could enormously greater values of entropy for systems dominated by gravitational interaction have its roots in the fact that gravity is significantly weaker as a force than the other three?
- perhaps: on account of this weakness, phase-space regions representing system as having larger momentum could be more easily accessible

AQFT-CST?

why no significant results? only recently, and those only for highly specialized spacetimes and matter states, or derivations of quantities not quite Bekenstein entropy:

1. Hollands and Ishibashi (2019)
2. D'Angelo (2021)
3. Kurpicz et al. (2021)

necessity of QSM?

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- in any event, even if we tried to use an SM based on QFT to calculate entropy of air in room, we couldn't do it without making approximations and idealizations that would essentially make all degrees of freedom "localized into something like classical atoms" anyway
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nature of spacetime and matter

- physical entropy has heretofore been attributed to *material* systems with non-trivial *dynamics*
- naively, spacetime geometry ("gravity") seems radically different from matter
- $\Rightarrow$ does black hole thermodynamics, and in particular gravitational entropy, militate in favor of effacing this difference?

one fundamental and characteristic property of "matter":

it possesses stress-energy
as represented by a T_{ab}

the thermodynamical fungibility of stress-energy

Ground of First Law of Thermodynamics: all forms of stress-energy are in principle ultimately fungible—any form of stress-energy can in principle be transformed into any other form

the family of all T_{ab} has a natural linear structure, and all stress-energy tensors must have the "physical dimension of stress-energy" $\Rightarrow$ the physical meaning of being able to add them together

“gravity has no stress-energy tensor,
so it can’t be matter”

but not so quick: sometimes possible to attribute *non-local* energy-like quantities, *i.e.*, not representable by a stress-energy tensor: gravitational radiation, ADM mass, various quasi-local masses, *etc.*—so this criterion is not so clear

anyway, other forms of energy in other theories are non-local (heat, work, Newtonian gravitational energy, ...)

another try – EFE contains only contributions from "matter" stress-energy, so in general relativity another "obvious" answer:

matter is Ricci tensor and gravity is Weyl tensor

BUT:

- in many spacetimes (*e.g.*, generalized FLRW: Carlo Alberto Mantica 2016) matter directly includes Weyl contributions
- and conditions on Weyl tensor determine form of the matter
- $\Rightarrow$ Weyl versus Ricci doesn't make the right distinction

BUT:

- in classical general relativity, "matter can transform into curvature" (gravitational collapse into a singularity)
- in black hole thermodynamics, "curvature can transform into matter" (Hawking radiation)

$\Longrightarrow$ breakdown of distinction between "matter" and "geometry"? requires radical changes to picture of ontology of spacetime and matter?

intriguing speculation: matter and geometry not truly independent, but different "manifestations of underlying unified entity" (compare electric and magnetic fields in Maxwell theory, time and space in special relativity, ...)

(seems to be suggested by some programs of quantum gravity)

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