Interaction and Evolution in Classical Mechanics and in Quantum Mechanics

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I've never worked on the measurement problem before. It always struck me as too hard. None of the extant "interpretations" of quantum mechanics suffices, to my mind, to deal with it. Indeed, I've tried my hardest to avoid anything to do with quantum mechanics my entire career. In my gut, I feel that there's something about the idea of "interaction" (superset of "measurement") we're getting fundamentally wrong, as with "simultaneity" before Einstein.

I will try to draw out and make precise some differences in the mathematical representations of "interaction" and "evolution" in classical mechanics and quantum mechanics, and begin to draw some conceptual lessons from that analysis.

I will *not* propose a "solution" to the measurement problem, rather a different way of looking at it and trying to approach it—an attempt to reconceptualize what a good answer to it might look like.

Outline

classical system

1. space of states $\mathcal{S}$ is even-dimensional manifold (or infinite)
2. evolution governed by Newton's Second Law (family of dynamical vector fields $\mathfrak{D}$)
3. "interaction": applying an "external force" to the system ("intervention")
4. has distinguished "free" evolution ("isolation" = no external force)

Newton's Second Law

free particle

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{0} \end{aligned} \tag{1}$$

dynamical vector field has components: $(\mathbf{v}, \mathbf{0})$

with interaction turned on

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{F}_{\text{stick}} \end{aligned} \tag{2}$$

dynamical vector field: $(\mathbf{v}, \mathbf{F}_{\text{stick}})$
interaction vector field: $(\mathbf{0}, \mathbf{F}_{\text{stick}})$

- dynamical evolution represented by vector field that always has $\mathbf{v}$ for the component of the $\frac{\partial}{\partial \mathbf{q}}$ part
- interaction represented by vector field that always has $\mathbf{0}$ for the component of the $\frac{\partial}{\partial \mathbf{q}}$ part: it "points only in velocital directions" (an acceleration, instantaneous change in velocity, not in position)
- any magnitude of force in any direction can be applied to any system (in principle)
- multiple simultaneously applied forces sum vectorially

Maxwell fields are classical systems too

free field

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\dot{\mathbf{B}} &= -\boldsymbol{\nabla} \times \mathbf{E} \\
\boldsymbol{\nabla} \cdot \mathbf{E} &= 0 \\
\dot{\mathbf{E}} &= \boldsymbol{\nabla} \times \mathbf{B}
\end{aligned}
$$

components of dynamical vector field: $(0,\, -\boldsymbol{\nabla} \times \mathbf{E},\, 0,\, \boldsymbol{\nabla} \times \mathbf{B})$

turn on interaction

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\dot{\mathbf{B}} &= -\boldsymbol{\nabla} \times \mathbf{E} \\
\boldsymbol{\nabla} \cdot \mathbf{E} &= \rho \\
\dot{\mathbf{E}} &= \boldsymbol{\nabla} \times \mathbf{B} + \boldsymbol{\jmath}
\end{aligned}
$$

dynamical vector field: $(0,\, -\boldsymbol{\nabla} \times \mathbf{E},\, \rho,\, \boldsymbol{\nabla} \times \mathbf{B} + \boldsymbol{\jmath})$
interaction vector field: $(0,\, \mathbf{0},\, \rho,\, \boldsymbol{\jmath})$

Even classical fields not in the Newtonian framework have the same structure!

Definition

A classical system *is fully characterized by:*

1. *space of states is even-dimensional manifold (or infinite);*
2. *the family of interaction vector fields $\mathfrak{I}$ has the structure of a vector space;*
3. *the family of dynamical vector fields $\mathfrak{D}$ has the structure of an affine space modeled on $\mathfrak{I}$.*

(the "fully characterized by" is justified by the theorem soon to come) “obvious” from formulation of Newton’s Second Law, Maxwell’s equations, and the representations of $\mathfrak{D}$ and $\mathfrak{I}$:

1. crudely, if I can hit a particle with two different sticks (“turn on” two separate electric charges/currents), I can hit it with (turn on) both at the same time
2. the difference between any two evolutions is just the force (electric charge/current) I need to apply to one to get to the other
3. this yields classification of quantities as “configurative” and “velocital”
4. free dynamical vector field distinguished by fact that only configurative quantities change

[one can make this all precise and rigorous, for finite systems and fields] conceptually, the family of possible interactions allows one to distinguish "configuration" from "velocity" quantities in a principled way:

- interactions directly modify the system's evolutions only in "velocital directions", *i.e.*, as generalized "accelerations" (why $\mathbf{B}$ is "configuration" in Lagrangian formulation of Maxwell theory, and $\mathbf{E}$ "velocity")
- correlatively, the physical meaning of "configuration" is that those quantities encode the possible interactions the system can have with its environment
- "velocity" quantities are always dynamical derivatives of "configuration" ones (in sense of affine structure on family of dynamical vector fields)

Lagrangian mechanics

Theorem (Curiel 1997/2014)

Given classical system with space of states $\mathcal{S}$ and families of vector fields $\mathfrak{D}$ and $\mathfrak{I}$:

1. *one can reconstruct the system's configuration space $\mathcal{C}$ from $\mathcal{S}$, based on the affine structure of $\mathfrak{D}$;*
2. *$\mathcal{S}$ is canonically isomorphic to $T\mathcal{C}$ (using the distinguished free dynamical vector field);*
3. *$\mathfrak{D}$ is then isomorphic to the family of vector fields on $T\mathcal{C}$ representing all solutions to the Euler-Lagrange equation ("second-order vector fields");*
4. *$\mathfrak{I}$ is isomorphic to the family of vector fields on $T\mathcal{C}$ representing generalized forces ("vertical vector fields", pointing straight up and down the fibers);*
5. *in particular, the vertical vector fields have the structure of a vector space, and the second-order vector fields the structure of an affine space modeled on it.*

the geometry of the Euler-Lagrange equation

Theorem (Curiel 1997/2014)

If a manifold $\mathcal{M}$ admits a formulation of the Euler-Lagrange equation, in the sense of having an operator mapping scalar fields to a family of vector fields $\mathfrak{D}$ having the appropriate structure ("integrable, complete almost-tangent structure"), then:

1. *it is diffeomorphic to a tangent bundle;*
2. *that operator allows one to construct configuration space $\mathcal{C}$, and to construct a canonical isomorphism between $\mathcal{M}$ and $T\mathcal{C}$ taking $\mathfrak{D}$ to the second-order vector fields;*
3. *in particular $\mathfrak{D}$ is an affine space modeled on the vector space of vector fields on $\mathcal{M}$ mapped to the vertical vector fields on $T\mathcal{C}$.*

Lagrangian mechanics defines and enforces:

1. configuration and velocity, and the difference between them
2. the fixed kinematical relation between them (the latter is the "dynamical derivative" of the former)
3. a notion of interaction ("interventions") distinct from dynamical evolution
4. a distinguished notion of "isolation"

if I know only the space of states as an abstract manifold and how to solve the Euler-Lagrange equation, I can reconstruct **everything else**

$\Rightarrow$ *the dynamics by itself automatically defines each of and encodes the difference between "configuration" and "velocity" and, correlatively, between "evolution" and "interaction", and uniquely determines "isolation"*

this is what I mean by "classical system":

1. mathematical and conceptual distinction between "configuration" and "velocity" encoded in the dynamics
2. mathematical and conceptual distinction between "evolution" and "interaction" encoded in the difference between configuration and velocity
3. and so a naturally distinguished "free" evolution

always a clean separation between a system's degrees of freedom and those of its environment—no matter the interaction, I can determine the system's evolution without knowing any details about the environment's degrees of freedom or its evolution

(akin to Don Howard's reconstruction of Bohr's notion of "classical", made rigorous; difficult to see how to analyze general relativity according to these ideas)

Interaction and Evolution in Newtonian and Lagrangian Mechanics

Classical Spacetime Structure

Interaction and Evolution in Hamiltonian Mechanics

Interaction and Evolution in Quantum Mechanics

Where Does This Leave Us?

in classical mechanics, evolution characterizes time and its geometry, while interaction (including "isolation") characterizes space and its geometry; both jointly determine the full 4-dimensional flat affine geometry

1. because the family of dynamical vector fields has the structure of an affine space, the dynamical evolutions (integral curves) have a natural affine parametrization
2. that determines the temporal metric: the ratios of the "lengths" of different intervals on the same curve is determined by the affine structure
3. the interactions determine configuration space; how to show that it encodes Euclidean $\mathbb{R}^3$?
4. at every point of configuration space, the tangent plane can be naturally identified with the free dynamical vector field (the system moving freely with all possible velocities with respect to the frame of reference naturally defined by that configuration, *viz.*, the stationarity of that configuration)
5. those velocity differences correspond to Galilean boosts, the family of which naturally has the structure of Euclidean $\mathbb{R}^3$ (once the zero point is fixed, which the frame of reference does)

6. the free vector field on the Newtonian space of states induces an affine connection on the induced configuration space: at each point of configuration space, each tangent vector ("velocity") determines a unique geodesic, that one that lifts to one of the integral curves of the free vector field on the space of states
7. this affine connection is flat
8. it projects down to a flat affine connection on the Euclidean $\mathbb{R}^3$ constructed previously, and is naturally identified with the canonical flat affine connection on Newtonian 3-space
9. Lie-deriving it with respect to the free evolution vector field yields the canonical flat affine connection on 4-dimensional Newtonian spacetime

(This is, in essence, the construction of Stein 1967, made precise and rigorous, capturing Newton's insight that the dynamical structure of his framework encodes the spatiotemporal structure required to formulate his laws.)

the distinction between time and space is built in to the structure of the dynamics, as is the geometry of spacetime itself

Theorem (Curiel 2019)

The Euclidean geometry of ordinary space, the metrical structure of time, and the flat affine geometry of 4-dimensional spacetime is entirely determined by the dynamical structure of a classical system.

Conjecture

All the affine structures induced on configuration space by a different choice of conservative second-order vector field are equivalent in the sense that they yield the same geometrized Newtonian gravity spacetime (i.e., in the same equivalence class in the sense of Trautman, related to each other by addition of an appropriate difference tensor to the derivative operator).

Hamiltonian mechanics

- phase space is a symplectic manifold
- family of dynamical vector fields $\mathfrak{D}$ is a Lie algebra, formed from solving Hamilton's equation for all Hamiltonians $(\boldsymbol{\xi} = \boldsymbol{\Omega}(\boldsymbol{\nabla} H))$
- only fixed relations between $\mathbf{q}$ and $\mathbf{p}$ are canonical Poisson brackets:

$$\begin{aligned} \{q_i, q_j\} &= 0 \\ \{q_i, p_j\} &= \delta_{ij} \\ \{p_i, p_j\} &= 0 \end{aligned} \tag{3}$$

- "interaction" is adding another Hamiltonian:

$\mathfrak{D}$ is same as $\mathfrak{I}$

Theorem (Curiel 2014)

Fix an even-dimensional, orientable manifold with a vector space of vector fields on it and a Poisson bracket structure. Then:

1. *the Poisson bracket arises from a symplectic structure,*
2. *and the vector space includes all and only solutions to Hamilton's equation formulated with it*

if and only if

1. *the vector space spans the tangent planes,*
2. *and the manifold has a family of coordinate systems whose coordinate functions satisfy the canonical Poisson bracket relations,*
3. *and the associated coordinate vector fields leave the vector space invariant under the action of the Lie bracket.*

Hamiltonian mechanics is not a classical framework:

1. the families of evolution and interaction vector fields are identical, having the structure of a Lie algebra (vector space)
2. there is no principled distinction between “configuration” and “momentum”; in particular the latter is not the dynamical derivative of the former, and their fixed relations are symmetric
3. *a fortiori*, there is no naturally distinguished “free evolution” vector field (the zero vector field is not a good candidate—too degenerate—one wants to allow “constant changes to configuration” for free evolution)

if I give you a manifold and tell you how to formulate Hamilton's equation on it (a symplectic structure)—if I tell you the structure of the dynamics, *i.e.*, the causal structure:

1. you can't tell me what's configuration and what's momentum ("space" versus temporal derivatives)
2. you can't distinguish interactions from evolutions ("interventions")
3. you can't tell what is free evolution ("isolation")

$\Longrightarrow$ **no clean separation between a system's degrees of freedom and those of its environment**

Why, then, does Hamiltonian mechanics work so well to model "classical" systems? Because

1. by fiat, we identify some variables as "momentum"
2. then restrict attention only to Hamiltonians having a special form, those "purely quadratic in momentum"
3. this reproduces the fixed kinematical relations between configuration and velocity in classical systems

examples:

- $H = \frac{1}{2}p^2 + \frac{1}{2}q^2$ is a simple harmonic oscillator, yielding the correct classical relation $v = \dot{q}$ under the Legendre transform
- think of $H = \frac{1}{2}p^2 + \frac{1}{2}q^2 + p$, however—that yields $v = \dot{q} + 1$, which is physically meaningless
- $H = p^{5/2}$ or $H = \sin(q)$ are even worse

the situation is essentially the same as for Hamiltonian mechanics:

1. dynamical vector fields (“evolutions”) are unitary flows on Hilbert space
2. unitary flows are exponentiated Hamiltonians (self-adjoint operators)
3. “interaction” is just adding another Hamiltonian
4. only fixed relation between “configuration” and “momentum” is canonical commutation relation

$$[Q,\, P] = i\hbar I$$

1. the families of evolution and interaction vector fields are identical, forming a C^*-algebra
2. there is no principled distinction between "configuration" and "momentum"; in particular the latter is not the dynamical derivative of the former, and their fixed relations are symmetric
3. *a fortiori*, there is no naturally distinguished "free evolution" vector field (the zero vector field is again not a good candidate)

if I give you a Hilbert space and its family of self-adjoint operators—the dynamics, *i.e.*, the causal structure:

1. you can't tell me what's configuration and what's momentum
2. you can't distinguish interactions from evolutions ("external interventions" such as measurements)
3. you can't tell what is free evolution ("isolation")

$\Longrightarrow$ **no clean separation between a system's degrees of freedom and those of its environment**

(akin to Don Howard's reconstruction of Bohr's understanding of entanglement, and his insistence on "complementarity" in quantum measurements, made rigorous)

How do we distinguish the Qs and the Ps in quantum mechanics, and so define a "free" evolution? Two ways:

1. by fiat
2. introduce representation of Poincaré or Galileian group *à la* Wigner, and define Q as the generator of translations, *etc*.

In both cases, we must explicitly and by hand hook the dynamics up to the background spacetime structure to get a principled distinction between configuration and momentum, and so define "free" and "interacting"

we don't get it from the dynamics alone
as in classical mechanics

WILD SPECULATION

perhaps the lesson is that there is at bottom no real difference between q and p in quantum mechanics

this would strongly suggest that quantum gravity should not be formulated in a 3+1 framework

I don't know

possible responses

1. modify quantum mechanics/look for interpretation that makes difference between "evolution" and "interaction" clear, as in classical mechanics—this is, implicitly, most common approach in most popular "interpretations" (many worlds with preferred basis, Bohm, GRW, . . . —but information-theoretic approaches?)
2. reconceptualize "interaction" in quantum mechanics so as to accord with these facts
3. get rid of the distinction between evolution and interaction altogether, and reconceptualize "dynamics"

(1) smells wrong to me—it doesn't take quantum mechanics seriously enough—like Einstein picking a preferred frame rather than reconceiving simultaneity; (2) may still be too conservative, given the experimental and mathematical facts

So maybe wait for our Einstein to implement (3)?