Classical Mechanics Is Lagrangian; It Is Not Hamiltonian

Erik Curiel

Rotman Institute of Philosophy
Department of Philosophy
University of Western Ontario

erik@strangebeautiful.com

Outline

1. Classical and Dynamical Systems

2. Possible Interactions and the Structure of the Space of States

3. Classical Mechanics Is Lagrangian

4. Classical Mechanics Is Not Hamiltonian

Atheoretical Adumbration of a Classical System

1. something one can interact with
2. bears quantities: magnitudes measurable by apparatuses exploiting particular couplings
3. in states: consistent aggregations of values for all its quantities, sufficient for id at a moment
4. evolves: changes state (in general) over time

Atheoretical Characterization of a Dynamical System

space of states set of identifiable states
quantities scalar/tensorial fields on space of states
evolutions family of vector fields ("kinematically possible")
isolation distinguished vector field ("free kinematical vector field")

representation of all physically significant structure

Physical Meaning of Elements of a Dynamical System

quantities individuate and identify states; define topology, differential structure on space of states

space of states arcwise-connected = all states of "same sys"; $\exists n$ (even or ∞), minimum quantities needed to individuate and identify states

evolutions "first-order differential equations appropriate for classical systems"

isolation "we know how to shield system"

Why "Atheoretical"

no claim or interpretation depends on fixation of framework or theory (e.g., no "configuration" or "momentum" distinguished)

reaches down to, represents structure at very deep level of our understanding of classical systems

Ends

1. recovery of intrinsic geometry of space of kinematically possible vector fields ("evolutions")
2. intrinsic construction of configuration space
3. intrinsic characterization of space of states' differential topology: tangent bundle of configuration space

Means

1. characterize family of possible interactions
2. intrinsic differentiation of configurative from velocital quantities
3. derive intrinsic geometry of family of possible interactions

Configurative Quantities?

- where does configuration space come from?
- why do we need it?
- $\mathbf{E}$ or $\mathbf{B}$ as configuration? why?

Example: Free Newtonian Particle

- parametrize space of states by $\mathbf{x}$ and $\mathbf{v}$ ("natural coordinates")
- equations of motion:

$$\dot{\mathbf{x}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = \mathbf{0}$$

- kinematical vector field: $(\mathbf{v}, \mathbf{0})$ ("free kinematical field")

Example: Hit Newtonian Particle with a Stick

- equations of motion:

$$\dot{\mathbf{x}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = \mathbf{F}_{\text{stick}}$$

- kinematical vector field: $(\mathbf{v}, \mathbf{F}_{\text{stick}})$ ("kinematically possible vector field")

  empirical observation*: velocities need not evolve continuously as interactions turn on and off—can happen as abruptly as one likes; position always evolves continuously*

Example: Free Electromagnetic Field

- parametrize space of states by $\boldsymbol{\nabla} \cdot \mathbf{B}$, $\dot{\mathbf{B}}$, $\boldsymbol{\nabla} \cdot \mathbf{E}$, $\dot{\mathbf{E}}$ ("natural coordinates")
- equations of motion (Maxwell's Equations):

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \dot{\mathbf{B}} &= -\boldsymbol{\nabla} \times \mathbf{E} \\ \boldsymbol{\nabla} \cdot \mathbf{E} &= 0 \\ \dot{\mathbf{E}} &= \boldsymbol{\nabla} \times \mathbf{B} \end{aligned}$$

- kinematical vector field: $(0,\ -\boldsymbol{\nabla} \times \mathbf{E},\ 0,\ \boldsymbol{\nabla} \times \mathbf{B})$ ("free kinematical field")

Example: Hit Electromagnetic Field with a Charged Stick

- equations of motion:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \dot{\mathbf{B}} &= -\boldsymbol{\nabla} \times \mathbf{E} \\ \boldsymbol{\nabla} \cdot \mathbf{E} &= \rho \\ \dot{\mathbf{E}} &= \boldsymbol{\nabla} \times \mathbf{B} + \boldsymbol{\jmath} \end{aligned}$$

- kinematical vector field: $(0,\ -\boldsymbol{\nabla} \times \mathbf{E},\ \rho,\ \boldsymbol{\nabla} \times \mathbf{B} + \boldsymbol{\jmath})$ ("kinematically possible vector field")

  empirical observation: *again, one half of quantities need not evolve continuously as interactions turn on and off—can happen as abruptly as one likes; other half always evolves continuously*

Configurative Quantities

A brute fact about the physical world: for all classical systems, only some physical quantities can evolve discontinuously ("those one can directly push around via allowed interactions"), whereas others always evolve continuously ("those one can't").

One generalizes: configurative quantities are those one cannot directly push around, and so are the ones that evolve continuously.

The Form of Allowed Interactions

difference of two kinematically possible vector fields always has special form,

- $(\mathbf{v},\, \mathbf{F}_2) - (\mathbf{v},\, \mathbf{F}_1) = (\mathbf{0},\, \mathbf{F}_2 - \mathbf{F}_1)$
- *e.g.*, difference for $\rho_1,\, \boldsymbol{\jmath}_1$ and $\rho_2,\, \boldsymbol{\jmath}_2$:

$$((0,\, \mathbf{0}),\, (\rho_2 - \rho_1,\, \boldsymbol{\jmath}_2 - \boldsymbol{\jmath}_1))$$

difference-vectors point only in "velocital directions", encode only rates of change for velocital quantities: accelerations

The Geometry of Allowed Interactions and Kinematically Possible Vector Fields

1. $(\mathbf{0},\, \mathbf{F}_1) + (\mathbf{0},\, \mathbf{F}_2)$ has same form: **vector space**
2. $(\mathbf{v},\, \mathbf{F}_1) + (\mathbf{0},\, \mathbf{F}_2)$ is kinematically possible vector field: **affine space** modeled on interactions

Configuration Space

1. divide space of states into equivalence classes: "is connected by an interaction vector field to"
2. by construction, all points in equivalence class have same configuration
3. space of equivalence classes is configuration space
4. in natural way, point of space of states becomes point of configuration space plus tangent vector at that point (= velocity)

Physical Meaning of Configuration Space

surprising

what counts as “configuration” for a classical system is not intrinsic to the system, but rather depends on the structure of its space of allowed interactions with other systems

The Space of States Is the Tangent Bundle of Configuration Space

1. fix point of configuration space
2. free kinematical vector field at that point takes all possible velocity values
3. this gives natural, one-to-one, onto mapping of space of states to tangent bundle of configuration space

The math starts to get hard.

Ends

Classical Systems Are Lagrangian structures of an abstract dynamical system, when pushed to the tangent bundle, allow one to construct a Lagrangian representation

Lagrangian Systems Are Classical a Lagrangian representation of a classical system, in the most minimal sense, allows one to construct its abstract dynamical representation

Means

exploit intrinsic geometry of tangent bundles

1. distinguished vector space of vector fields, "vertical vectors", possible generalized forces in Lagrangian mechanics (encoded in fiber-bundle structure)
2. distinguished affine space of vector fields modeled on vertical vector fields, possible solutions to Euler-Lagrange equation (encodes almost-tangent structure, $J^a{}_b$)
3. distinguished vertical vector field, "Liouville vector field" (Λ^a, encoded by almost-tangent structure)

The Euler-Lagrange Equation

To formulate Euler-Lagrange equation in invariant, geometrical terms,

$$2J^m{}_a\xi^n\nabla_{[m}(J^s{}_{n]}\nabla_s L) - J^m{}_a\nabla_m(\Lambda^n\nabla_n L - L) = 0$$

one needs (and only needs):

1. almost-tangent structure, $J^a{}_b$
2. Liouville vector field, Λ^a
3. Lagrangian L

Constructing the Euler-Lagrange Equation from Dynamical System Structure (Classical Systems Are Lagrangian)

using canonical isomorphism of dynamical space of states with tangent bundle over configuration space:

1. push kinematical vector fields over $\Rightarrow$ affine space of Lagrangian vector fields (same physical meaning)
2. push interaction vector fields over $\Rightarrow$ vector space of vertical vector fields (same physical meaning)
3. $\Longrightarrow J^a{}_b$ and Λ^a

Constructing Dynamical System Structure from the Euler-Lagrange Equation

an Euler-Lagrange operator is a (non-linear) functional that takes a scalar field to its associated Lagrangian vector field on a manifold that supports the formulation of the Euler-Lagrange equation

Theorem

A manifold has an Euler-Lagrange operator if and only if it is a tangent bundle over another space; the operator's action allows one to recover the space over which it is the tangent bundle.

Lagrangian Systems Are Classical

using canonical isomorphism of tangent bundle over configuration space with dynamical space of states:

1. push affine space of Lagrangian vector fields over $\Rightarrow$ kinematical vector fields
2. push vertical vector fields over $\Rightarrow$ vector space of interaction vector fields

Hamilton's Equation

To formulate Hamilton's equation in invariant, geometrical terms,

$$\Omega^{an}\nabla_n H = \xi^a$$

one needs (and only needs):

1. fixed, canonical symplectic structure, Ω^{ab}
2. Hamiltonian H

Geometry of Hamiltonian Mechanics

space of states any symplectic manifold (not necessarily isomorphic to tangent bundle)

space of dynamical evolutions Lie algebra (not isomorphic to affine space)

space of interaction vector fields same as space of Hamiltonian vector fields (not basis for affine space)

classical mechanics is not Hamiltonian;
Hamiltonian mechanics is not classical