Chicago Lectures in Physics

Mathematical Physics

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Introduction

One sometimes hears expressed the view that some sort of uncertainty principle operates in the interaction between mathematics and physics: the greater the mathematical care used to formulate a concept, the less the physical insight to be gained from that formulation. It is not difficult to imagine how such a viewpoint could come to be popular. It is often the case that the essential physical ideas of a discussion are smothered by mathematics through excessive definitions, concern over irrelevant generality, etc. Nonetheless, one can make a case that mathematics as mathematics, if used thoughtfully, is almost always useful—and occasionally essential—to progress in theoretical physics.

What one often tries to do in mathematics is to isolate some given structure for concentrated, individual study: what constructions, what results, what definitions, what relationships are available in the presence of a certain mathematical structure—and only that structure? But this is exactly the sort of thing that can be useful in physics, for, in a given physical application, some particular mathematical structure becomes available naturally, namely, that which arises from the physics of the problem. Thus mathematics can serve to provide a framework within which one deals only with quantities of physical significance, ignoring other, irrelevant things. One becomes able to focus on the physics. The idea is to isolate mathematical structures, one at a time, to learn what they are and what they can do. Such a body of knowledge, once established, can then be called upon whenever it makes contact with the physics.

An everyday example of this point is the idea of a derivative. One could imagine physicists who do not understand, as mathematics, the notion of a derivative and the properties of derivatives. Such physicists could still formulate physical laws, for example, by speaking of the "rate of change of . . . with . . ." They could use their physical intuition to obtain, as needed in various applications, particular properties of these "rates of change." It would be more convenient, however, to isolate the notion "derivative" once and for all, without direct reference to later physical applications of this concept. One learns what a derivative is and what its properties are: the geometrical significance of a derivative, the rule for taking the derivative of a product, etc. This established body of knowledge then comes into play automatically when the physics requires the use of derivatives. Having mastered the abstract concept "rate of change" all by itself, the mind is freed for the important, that is, the physical, issues.

The only problem is that it takes a certain amount of effort to learn mathematics. Fortunately, two circumstances here intervene. First, the mathematics one needs for theoretical physics can often be mastered simply by making a sufficient effort. This activity is quite different from, and far more straightforward than, the originality and creativity needed in physics itself. Second, it seems to be the case in practice that the mathematics one needs in physics is not of a highly sophisticated sort. One hardly ever uses elaborate theorems or long strings of definitions. Rather, what one almost always uses, in various areas of mathematics, is the five or six basic definitions, some examples to give the definitions life, a few lemmas to relate various definitions to each other, and a couple of constructions. In short, what one needs from mathematics is a general idea of what areas of mathematics are available and, in each area, enough of the flavor of what is going on to feel comfortable. This broad and largely shallow coverage should in my view be the stuff of "mathematical physics."

There is, of course, a second, more familiar role of mathematics in physics: that of solving specific physical problems which have already been formulated mathematically. This role encompasses such topics as special functions and solutions of differential equations. This second role has come to dominate the first in the traditional undergraduate and graduate curricula. My purpose, in part, is to argue for redressing the balance.

We shall here take a brief walking tour through various areas of mathematics, providing, where appropriate and available, examples in which this mathematics provides a framework for the formulation of physical ideas.

By way of general organization, chapters 2–24 deal with things algebraic and chapters 25–42 with things topological. In chapters 43–50 we discuss some special topics: structures which combine algebra and topology, Lebesque integrals, Hilbert spaces. Lest the impression be left that no difficult mathematics can ever be useful in physics, we provide, in chapters 51–56, a counterexample: the spectral theorem. Strictly speaking, the only prerequisites are a little elementary set theory, algebra, and, in a few places, some elementary calculus. Yet some informal contact with such objects as groups, vector spaces, and topological spaces would be most helpful.

The following texts are recommended for additional reading: A. H. Wallace, *Algebraic Topology* (Elmsford, NY: Pergamon, 1963), and C. Goffman and G. Pedrick, *First Course in Functional Analysis* (Englewood Cliffs, NJ: Prentice-Hall, 1965). Two examples of more advanced texts, to which the present text might be regarded as an introduction, are: M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (New York: Academic, 1972), and Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, *Analysis, Manifolds and Physics* (Amsterdam: North-Holland, 1982).

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Categories

In each area of mathematics (e.g., groups, topological spaces) there are available many definitions and constructions. It turns out, however, that there are a number of notions (e.g., that of a product) that occur naturally in various areas of mathematics, with only slight changes from one area to another. It is convenient to take advantage of this observation. Category theory can be described as that branch of mathematics in which one studies certain definitions in a broader context—without reference to the particular area to which the definition might be applied. It is the "mathematics of mathematics." Although this subject takes a little getting used to, it is, in my opinion, worth the effort. It provides a systematic framework that can help one to remember definitions in various areas of mathematics, to understand what many constructions mean and how they can be used, and even to invent useful definitions when needed. We here summarize a few facts from category theory.

A *category* consists of three things—i) a class O (whose elements will be called *objects*), ii) a set $\mathrm{Mor}(A,B)$ (whose elements will be called *morphisms* from A to B), where A and B are any two[1] objects, and iii) a rule which assigns, given any objects A, B, and C and any morphism φ from A to B and morphism ψ from B to C, a morphism, written $\psi \circ \varphi$, from A to C (this $\psi \circ \varphi$ will be called the *composition* of φ with ψ)—subject to the following two conditions:

1. Composition is associative. If A, B, C, and D are any four objects, and φ, ψ, and λ are morphisms from A to B, from B to C, and from C to D, respectively, then

$$(\lambda \circ \psi) \circ \varphi = \lambda \circ (\psi \circ \varphi) \ .$$

(Note that each side of this equation is a morphism from A to D.)

2. Identities exist. For each object A, there is a morphism i_A from A to A (called the *identity* morphism on A) with the following property: if φ is any morphism from A to B, then

$$\varphi \circ i_A = \varphi \ ;$$

if μ is any morphism from C to A, then

1. Here and hereafter, "two elements" means "two elements in a specific order," or, more formally, an "ordered pair."